The Clark Model for Business Cycle Analysis with Markov Switching Regimes

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Abstract

The Clark model is an unobserved components model consisting of a trend and a cycle components. The former is a random walk with stochastic drift where the effect of shocks is permanent, while the latter component is driven by a stationary process characterized by the transitory effect of shocks. In Clark's model the effect of shocks is symmetric throughout the phases of the cycle. Stylized facts reveal the presence of asymmetries where long and smooth expansion periods alternate with sharper and shorter recession periods. In order to account for these asymmetries, researches have drawn attention to models where changes of regime are modelled endogenously as a Markov process. In this thesis, we set up a general framework that encompasses reference models analyzed in the literature. We adopt Clark's structural time series model and extend it allowing for Markov switching regimes in the components and the parameters of the model. We illustrate the contribution from dynamic econometric models to the empirical analysis of gross domestic product time series. In a first stage, the analysis is carried out separately for a trend-cycle unobserved components model and a Markov switching regime model. Next, we conduct two applications to compare results based on exact and approximate maximum likelihood in the context of the Clark model with Markov switching regimes. Upon this framework, we analyze further empirical issues that have been addressed separately in the literature: measure the persistence of shocks and estimation of the contemporaneous correlation between components. Finally, the presence of two transitory components is discussed in a model with asymmetries in the trend.

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1 Introduction

The business cycle has long been studied in economic theory and empirical works. The work by Burns and Mitchell (1946) is widely cited as a pioneering research in the field. Broadly speaking, the analysis of the business cycle involves the detection of turning points indicating the beginning and the end of recession and expansion periods. The analysis is carried out upon observation of a macroeconomic variable measuring economic activity, e.g., gross domestic product (GDP). Bry and Boschan (1971) developed a procedure for the detection of turning points in a smooth signal. The procedure also enforces certain censoring rules related to the minimum duration of the phases. More recently, Harding and Pagan (2002) developed a procedure for the location of local maxima and minima subject to censoring rules specified by the user. The *dissection* of the cycle as proposed in the reference paper provides a descriptive analysis of the business cycle: duration, amplitude and strength of the phases as well as certain type of asymmetries measured as the ratios between the duration and amplitude of consecutive phases.

The searching algorithms mentioned above rely on a smooth input series often obtained by means of a filtering procedure. Some of these filters, such as the Hodrick-Prescott and the Baxter-King, are well known in economics, see Hodrick and Prescott (1997); Baxter and King (1999); Kaiser and Maravall (2001). Other filters known in several fields of engineering as Butterworth filters are used for the analysis of economic data as well, see Pollock (2000); Gómez (2001).

Beveridge and Nelson (1981) devised a procedure for the decomposition of a non-stationary time series into a permanent and a transitory component. The components are obtained upon the selection of an autoregressive moving average model for an integrated process (ARIMA). The interpretation of the Beveridge-Nelson (BN) decomposition is a matter of debate. See Morley (2007) and references therein for a survey on this topic. One interpretation regards the BN trend as an estimate of the trend in an unobserved component model. Another interpretation considers the BN trend as the permanent component of an integrated process. According to the first interpretation, the BN trend is an estimate and, hence, an unobserved component. In the second explanation, the interpretation of the BN trend is attached to the definition of an integrated process and is considered observable. The Wiener-Kolmogorov theory for the decomposition of an ARIMA model selected for the observed data has been shown to be useful for a wider range of practical purposes. See Burman (1980); Gómez and Maravall (2001) for an application of the methodology in the context of seasonal adjustment.

Clark (1987) proposed a formal econometric specification for a permanent (trend) plus transitory (cycle) model in an unobserved components framework. Given a state space representation of the model, it can be estimated by maximum likelihood by means of the Kalman filter as explained in

Harvey (1989). Harvey and Jaeger (1993) explore empirically the performance of different detrending techniques. They find some limitations in the Hodrick-Prescott filter and ARIMA modelling techniques and claim that the modelization by components; namely, trend, cycle, seasonal and irregular, is a convenient approach to capture the stylized facts observed in macroeconomic time series. In Clark's model, the trend component is a random walk with stochastic drift and the transitory component is modelled as a stationary autoregressive process. For the sake of clarity, it is worth making at this point a remark about what is usually understood by *trend component* and *cyclical component*. The former refers to the long-run evolution of the data, whereas the latter is typically related to cycles that are 1.5-12 years long, following Burns and Mitchell (1946). In a Fourier decomposition of the series, the trend component is represented by cycles of frequencies close to zero. Real data do not contain a whole trend cycle (the period of a cycle of frequency zero is infinite). We can observe a few cycles of frequencies related to the business cycle.

In Clark's model the effect of shocks is symmetric throughout the phases of the cycle. Nowadays there are two major *stylized facts* broadly accepted: 1) the business cycle consists of relatively long and smooth expansion periods alternating with sharper and shorter recession periods, 2) the amplitude of a recession is correlated with the following expansion, while the amplitude of an expansion is uncorrelated with the amplitude of the subsequent contraction. In order to account for these asymmetries, researches –following the work in Hamilton (1989)– have drawn attention to models where changes of regime are modelled endogenously as a Markov process. In this thesis, we extend the traditional structural time series model advocated in Clark (1987) considering Markov switching regimes. Following this direction, two major models are found in the literature: the generalized Hamilton model, Lam (1990), and Friedman's plucking model, Kim and Nelson (1999*a*). We set up a general framework that encompasses both models. We adopt Clark's structural time series model and include the possibility of asymmetries in the components. The models of the reference papers cited above are extended with Markov switching parameters.

Typically, the components in a structural time series model are assumed to be uncorrelated. This assumption achieves identification of the parameters of the model. In the context of Clark's model, Morley et al. (2003) show that, under certain conditions, the restriction of no correlation between the disturbance terms in the permanent and transitory components is not necessary for identification. Sinclair (2007) estimates the covariance between components in the permanent plus transitory model with asymmetries in the transitory component. We will explore this issue in a model with asymmetries in the trend and Markov switching variances and covariance.

The remaining of the thesis is organized as follows. Section 2 introduces unobserved components models. Markov Switching models are introduced in Section 3. The reference setting in this thesis combines the previous two methodologies. Our benchmark model and the estimation procedure is given in Section 4. Empirical results are described in Section 5. Section 6 concludes.

2 Unobserved Components Models

In this section we introduce Clark's unobserved components model for business cycle analysis and discuss maximum likelihood estimation.

2.1 A Permanent plus Transitory Components Model

Structural time series models and state space methods are broadly used for the decomposition of time series into unobserved components. In the spirit of the model proposed in Clark (1987), we define a general model that consists of a permanent component (trend) and a transitory component (cycle). The model is specified as follows:

$$y_t = n_t + x_t + u_t\,, \qquad u_t \sim NID(0, \sigma_u^2)\,, \tag{1}$$

$$n_t = g_{t-1} + n_{t-1} + v_t\,, \qquad v_t \sim NID(0, \sigma_v^2)\,, \tag{2}$$

$$g_t = g_{t-1} + w_t\,, \qquad w_t \sim NID(0, \sigma_w^2)\,, \tag{3}$$

$$x_t = \sum_{i=1}^{k} \phi_i x_{t-i} + e_t\,, \qquad e_t \sim NID(0, \sigma_e^2)\,, \tag{4}$$

for $t = 1, 2, ...n$. The roots of the polynomial $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_k L^k$, where L is the lag operator and k is the AR lag order, lie outside the unit circle. Typically, the disturbance terms, u_t, v_t, w_t and e_t, are considered mutually uncorrelated. We will further discuss this issue shortly.

The trend component, n_t, is a random walk with drift, where the drift follows in turn a random walk. If σ_w^2 is set equal to zero, the trend is a random walk with deterministic drift equal to g_0. When both $\sigma_v^2 = \sigma_w^2 = 0$, the component n_t is a deterministic intercept equal to $n_0 + g_0$ for all t. The cyclical component, x_t, is a stationary autoregressive (AR) process with periodicity determined by the roots of the polynomial $\phi(L)$. Given a possibly complex root $r = a + bi$, with $i = \sqrt{-1}$, the modulus of the root is $m = \sqrt{a^2 + b^2}$. The frequency of the cycle related to the polynomial $\phi(L)$ is $\arccos(1/m)$. Alternatively, a trigonometric specification based on sine-cosine waves can be defined as:

$$x_t = \rho\,(\alpha \cos \lambda\, t + \beta \sin \lambda\, t) + e_t\,.$$

The amplitude of the waves is $\sqrt{\alpha^2 + \beta^2}$ and the phase is $\tan^{-1} \beta/\alpha$. The parameter ρ lies within the $(0, 1)$ interval. Following Harvey (1989), the trigonometric representation given above can be written as an AR(1) process as follows:

$$x_t = \rho \left(x_{t-1} \cos\lambda + x^*_{t-1} \sin\lambda \right) + e_t\,, \qquad e_t \sim NID(0, \sigma^2_e) \tag{5}$$

$$x^*_t = \rho \left(-x_{t-1} \sin\lambda + x^*_{t-1} \cos\lambda \right) + e^*_t\,, \qquad e^*_t \sim NID(0, \sigma^{2^*}_e)\,, \tag{6}$$

where λ is the frequency of the cycle with period $2\pi/\lambda$. The term x^*_t appears only for the specification of the cyclical component as a recursive process and is not relevant in itself. We will consider $\sigma^2_e = \sigma^{2^*}_e$. The state space representation of the structural model introduced in this section is given in Appendix A.1.

2.2 Identification of Model Parameters

It is illuminating to rewrite the model in equations (1)-(4), for instance with $k = 2$, taking the stationarity operator, $(1-L)^2 \equiv \Delta^2$, and multiplying both sides by $\phi(L) = (1 - \phi_1 L - \phi_2 L^2)$. For simplicity, we omit the disturbance term, u_t, in the observation equation:

$$\phi(L)\Delta^2 y_t = \underbrace{\phi(L) w_{t-1} + \phi(L)\Delta v_t + \Delta^2 e_t}_{\epsilon_t + \theta_1\,\epsilon_{t-1} + \theta_2\,\epsilon_{t-2} + \theta_3\,\epsilon_{t-3}}\,. \tag{7}$$

The previous expression is known as the reduced form of the structural model and is an autoregressive (AR) moving average (MA) process of orders 2 and 3, respectively for the AR and MA parts, for a second order integrated time series, I(2): ARIMA(2,2,3) model. The lagged disturbance terms in the right hand side of equation (7) can be written as an MA(3) process for a single random variable, ϵ_t, by Granger's representation lemma.

The unobserved components model and the ARIMA model have the same autocovariance structure. Thus, equating the non-zero autocovariances from both representations, a relationship between the parameters in both models is obtained. Despite the existence of a relationship between the parameters of both equivalent models, a solution to the system of equations is not feasible in this case. It can be shown that in the model above there are four non-zero autocovariances. Hence, there are four equations for six unknown parameters in the structural model: $\sigma^2_v, \sigma^2_w, \sigma^2_e, \sigma_{vw}, \sigma_{ve}$ and σ_{we}. The parameters of the structural model are not identified. A common identifying assumption is that the innovations in the structural model are mutually uncorrelated, being the covariances $\sigma_{v,w}$, $\sigma_{v,e}$ and $\sigma_{w,e}$ equal to zero. Notice that this assumption leads to three unknown parameters (the variances) for four non-zero autocovariances and, hence, the model is overidentified.

Similarly, it can be shown that the corresponding reduced form for the model where the trend follows a random walk with deterministic drift ($\sigma^2_w = 0$) is an ARIMA(2,1,2) model. Furthermore, this structural model with $\sigma_{we} \neq 0$ is exactly identified. See Morley et al. (2003) for details.

2.3 Maximum Likelihood Estimation

The linear model given in equations (1)-(4) can be estimated by maximum likelihood (ML) using the Kalman filter. The likelihood of the set of parameters $\theta = (\sigma_v^2, \sigma_w^2, \sigma_e^2, \phi_1, \phi_2)$, given the vector of observed data, y, is:

$$L(\theta|y) = \prod_{t=1}^{n} \left[\frac{1}{\sqrt{2\pi H_t}} \exp\left(-\frac{\eta_t^2}{2H_t} \right) \right] , \tag{8}$$

where H_t is the variance of the prediction error η_t defined below. The likelihood function can be maximized to obtain estimates of the parameters. In practice, the logarithm of the likelihood is optimized for computational stability, among other reasons.

The Kalman filter is a recursive procedure that allows us to compute the contribution of each observation to the likelihood. In the final step we get the likelihood of the parameters of the model given the entire data set. The classical approach assumes that the sample data are distributed according to a Normal distribution. The expected mean for an observation at time t is the forecast of that observation conditional on all the past information with the variance of the corresponding prediction error. We use the following notation to state this point: $y_t \sim N\left(y_{t|t-1}, H_t\right)$, where H_t is the variance of $\eta_t = y_t - y_{t|t-1}$.

Given the true values for the parameters of the Normal distribution, the expectation for the next period can be computed. In practice, we do not know those parameters that identify the distribution and need to be estimated. For a Normal distribution, the goal is to estimate its mean and variance. The Kalman filter computes expected values of the unobserved components from $t = 1, 2, ..., n$ together with the prediction error and its variance. They are updated as a new observation is made available at each iteration. Following the equations given below, the procedure of the algorithm can be implemented for a wide range of models for which a state space representation exists. A state space representation of a general dynamic linear model consists of a measurement and a transition equation:

$$\begin{aligned} &\text{Measurement equation:} \quad && y_t = Z\alpha_t + \epsilon_t \,, \\ &\text{Transition equation:} && \alpha_t = F\alpha_{t-1} + \zeta_t \,, \end{aligned} \tag{9}$$

where

$$\begin{pmatrix} \epsilon_t \\ \zeta_t \end{pmatrix} \sim NID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} , \begin{pmatrix} R & 0 \\ 0 & Q \end{pmatrix} \right) .$$

The state vector α_t is of dimenstion $K \times 1$ and the matrix F is of dimension $K \times K$. The Kalman filter iterates the following equations for $t = 1, 2, ..., n$:

Forecast

$$
\begin{aligned}
&\alpha_{t|t-1} = F\alpha_{t-1|t-1}\,, \quad \text{expectation conditional on information up to time } t-1.\\
&P_{t|t-1} = FP_{t-1|t-1}F' + Q\,, \quad \text{conditional variance covariance matrix of } \alpha_{t|t-1}.\\
&\eta_{t|t-1} = y_t - Z\alpha_{t|t-1}\,, \quad \text{prediction error.}\\
&H_{t|t-1} = ZP_{t|t-1}Z' + R\,, \quad \text{conditional variance of the prediction error } \eta_{t|t-1}.\\
&K_t = P_{t|t-1}Z'H_{t|t-1}^{-1}\,, \quad \text{Kalman gain.}
\end{aligned}
\tag{10}
$$

Update

$$
\begin{aligned}
&\alpha_{t|t} = \alpha_{t|t-1} + K_t\eta_{t|t-1}\,, \quad \text{expectation conditional on information up to time } t.\\
&P_{t|t} = P_{t|t-1} - K_tZP_{t|t-1}\,, \quad \text{conditional variance covariance matrix of } \alpha_{t|t}.
\end{aligned}
$$

Initial values for time $t = 0$ are required to start the recursions. If relevant information is available, the practitioner may propose some tentative values for starting the filter. In order to avoid the effect of this choice, the contributions to the likelihood of the first observations can be omitted. Alternatively, the initial values can be included in the set of parameters to be estimated by ML together with the remaining parameters of the model. A state space representation for our reference model is given in Appendix A.1.

The likelihood function –evaluated by means of the Kalman filter– can be optimized by numerical optimization given a initial set of values for the parameters of the model. We use the BFGS (Broyden, Fletcher, Goldfarb and Shanno) numerical optimization algorithm.

Finally, an estimate of the state vector vector and its covariance matrix conditional on the whole sample information can be obtained as follows:

$$
\begin{aligned}
\alpha_{t|T} &= \alpha_{t|t} + P_{t|t}F'P_{t+1|t}^{-1}\left(\alpha_{t+1|T} - F\alpha_{t|t}\right)\,,\\
P_{t|T} &= P_{t|t} + P_{t|t}F'P_{t+1|t}^{-1}\left(P_{t+1|T} - P_{t+1|t}\right){P_{t+1|t}^{-1}}'FP_{t|t}'\,,
\end{aligned}
$$

for $t = n-1, n-2, ..., 1$. These equations are known as the smoothing equations. The initial values for the smoothing, $t = n$, are obtained from the last iteration of the filtering equations in (10).

3 Markov Switching Models

In this section we introduce a non-linear autoregressive model where the mean undergoes changes of regime. The transitions from one regime to another are modelled as a first order Markov switching process. The model introduced in this section is intended to capture asymmetries in the effect of shocks from each regime. It is especially relevant for the analysis of the business cycle, since the stylized facts indicate the presence of asymmetries between recession periods (short and strong) and expansion periods (longer and smoother).

Hamilton (1989) proposed the following autoregressive process with switching mean for the growth rates of the gross domestic product:

$$y_t - \mu_{S_t} = \sum_{l=1}^{p} \phi_l (y_{t-l} - \mu_{S_{t-l}}) + \epsilon_t \,, \quad \epsilon_t \sim NID(0, \sigma^2) \,, \tag{11}$$

$$\mu_{S_t} = \mu_1 S_{1t} + \mu_2 S_{2t} \,,$$

where $S_{jt} = 1$ if $S_t = j$ and $S_{jt} = 0$ otherwise and S_t is a first order Markov process with transition probabilites:

$$Pr(S_t = j | S_{t-1} = i) = p_{ij} \,, \quad i, j = 1, 2 \,.$$

The roots of the polynomial $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$ lie outside the unit circle. Although the model above can be cast in state space form, as it is non-linear the Kalman filter cannot be used to compute the likelihood function. Forward-backward type algorithms have been developed in several fields of engineering for the estimation of hidden Markov models. Hamilton (1989) developed an algorithm for the estimation of Markov switching models by maximum likelihood. A particularity of the Hamilton filtering algorithm is that it deals with serial correlation. In particular, the likelihood of a general AR model of order p with Markov switching parameters can be computed. Time dependent data are common in macroeconomic time series and lagged dependent variables are often included as explanatory variables. In our case, a general AR model of order p will be estimated.

In this context, maximum likelihood estimation involves two main steps that are iterated for the entire sample from $t = 1, 2, ..., n$ given some initial values for $t = 0$: 1) computation of the conditional densities of the data given the parameters at each regime; 2) computation of the filtered probabilities of each regime. As we will see, the likelihood function is a weighted average of the conditional densities where the weighting factors are the probabilities obtained in a second step.

The joint density of the contemporaneous data y_t and the Markov process S_t along with p lags conditional on information up to time $t-1$ is given by:

$$\begin{aligned} & f\left(y_t, S_t, S_{t-1}, ..., S_{t-p} | \psi_{t-1}\right) \\ & = f\left(y_t | S_t, S_{t-1}, ..., S_{t-p}\right) \times Pr\left(S_t, S_{t-1}, ..., S_{t-p} | \psi_{t-1}\right) \,. \end{aligned}$$

The marginal distribution of the data can be obtained integrating the vector $(S_t, S_{t-1}, ..., S_{t-p})$ out of the joint density:

$$
\begin{aligned}
f\left(y_t|\psi_{t-1}\right) &= \sum_{S_t=1}^{2}\sum_{S_{t-1}=1}^{2}\cdots\sum_{S_{t-p}=1}^{2} f\left(y_t, S_t, S_{t-1}, ...S_{t-p}|\psi_{t-1}\right) \qquad (12)\\
&= \sum_{S_t=1}^{2}\sum_{S_{t-1}=1}^{2}\cdots\sum_{S_{t-p}=1}^{2} f\left(y_t|S_t, S_{t-1}, ...S_{t-p}, \psi_{t-1}\right)\\
&\times \;\; Pr\left(S_t, S_{t-1}, ...S_{t-p}|\psi_{t-1}\right)\,.
\end{aligned}
$$

We can see that the marginal distribution of the data given the past information is a weighted average of 2^{p+1} conditional densities. Each conditional density is related to a possible path of the Markov process S_t from period t to $t-p$.

Under the assumption of Gaussian disturbances ϵ_t, the conditional distribution takes the expression:

$$
\begin{aligned}
&f\left(y_t|S_t, S_{t-1}, ...S_{t-p}, \psi_{t-1}\right)\\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\left((y_t-\mu_{S_t})-\sum_{l=1}^{p}\phi_l(y_{t-l}-\mu_{S_{t-l}})\right)^2}{2\sigma^2}\right)\,.
\end{aligned}
$$

The log-likelihood function is obtained adding the contributions given in equation (12) for all the observations:

$$
\text{log-likelihood} = \sum_{t=1}^{n}\log\, f\left(y_t|\psi_{t-1}\right)
$$

and can then be maximized for the set of parameters $\theta = (p_{11}, p_{22}, \mu_1, \mu_2, \phi_l, \sigma^2)$.

For the maximum likelihood procedure to be feasible, the weighting terms $Pr\left(S_t, S_{t-1}, ..., S_{t-p}|\psi_{t-1}\right)$ in equation (12) need to be computed. The filtering algorithm developed in Hamilton (1989) to this end is described in Appendix A.2. Initialization of the filter can be done using the steady state probabilities as indicated in the Appendix. If the filter is run for evaluation of the likelihood function and maximum likelihood estimation, the optimization algorithm must be started with a set of initial parameter values (among them p_{11} and p_{22}) proposed by the practitioner.

Markov switching models have been shown to be useful for the analysis of macroeconomic data: an interpretation relevant for the economic analysis may be often attached to each regime, for instance as recession and expansion periods; Kaufmann (2002) investigates on the presence of a potential asymmetric effect in the monetary policy over the business cycle; Chauvet et al. (2002) study the dynamic of unemployment and the relationship with the phases of the business cycle. For a review of further empirical research in this field see volume 27 issue number 2 published in *Empirical Economics* in the year 2002.

4 Structural Time Series Models with Markov Switching Regimes

The combination of the techniques described in previous sections leads us to the ultimate aim of this thesis: specification and implementation of a procedure for the estimation of a structural time series model with a permanent trend component and a transitory cyclical component subject to changes in regime. The model in equations (1)-(4) is extended accounting for asymmetries in the components through a Markov switching variable as well as changes in regime in the variances of the innovation terms. There is precedent in the literature for these specifications, see Kim and Nelson (1999*b*) for a review. The state space representation that we follow for the general model is given in Appendix A.1. This representation provides a unified approach for the reference models analyzed in the literature as well as further versions that we explore below.

4.1 Restricted Versions of the Clark Model with Markov Switching Regimes

The implementation of the general setting employed in this thesis encompasses two particular models found in the literature: Lam's and Friedman's Plucking model, as well as other extensions required for further empirical questions that we will address in Section 5.

4.1.1 Lam's generalized Hamilton model

The model advocated in Lam (1990) is a generalized version of Hamilton's model, see Hamilton (1989). The original paper uses the first differences of the data as the dependent variable and a representation different from the state space form given in Appendix A.1. In our general structural model, we write Lam's model as a trend component modelled as a random walk with a Markov switching, τ_{S_t}, plus a transitory component, x_t:

$$y_t = n_t + x_t + u_t\,, \qquad u_t \sim NID(0, \sigma_u^2) \tag{13}$$

$$n_t = n_{t-1} + \tau_0 + \tau_1 S_t + v_t\,, \qquad v_t \sim NID(0, \sigma_v^2) \tag{14}$$

$$x_t = \sum_{i=1}^{k} \phi_i x_{t-i} + e_t\,, \qquad e_t \sim NID(0, \sigma_e^2) \tag{15}$$

where S_t is an indicator variable for the regime governing the trend component. It takes the value 0 for the first regime and 1 for the second regime and is modelled as a first order Markov process with probabilities $Pr(S_t = j|S_{t-1} = i) = p_{ij}$ for $i, j = 1, 2$. The original model does not include a disturbance term in the trend component, v_t, since the dependent variable is the first differences of the data. The innovation u_t in the observation equation is neither considered there. In our representation, to account for a larger variance in the non-differenced series, we will include at least one of these disturbance terms besides the disturbance in the transitory component, e_t.

In the generalized version of Hamilton's model the entire history of the Markov process S_t is involved in each state observation at time t. This fact is pointed out in Lam (1990) by rewriting the model in the stationary form:

$$\Delta y_t = \tau_0 + \tau_1 S_t + \Delta x_t \,.$$

Solving for x_t by backwards substitutions yields

$$x_t = \sum_{t=1}^{t} y_t - \tau_0 t - \tau_1 \sum_{t=1}^{t} S_t + x_0 \,,$$

where we can see that the sum of the Markov process over all periods up to time t shows up in the expression for the cyclical component at time t, x_t. The algorithm for the estimation of the model by maximum likelihood has to deal with this issue. Hamilton's algorithm involves the joint distribution of the series at time t and the contemporaneous and k lagged observations of the Markov process, instead. Lam (1990) develops an algorithm where, treating $\sum_{t=1}^{t} S_t$ as an additional state variable, exact maximum likelihood can be carried out.

Both Hamilton's and Lam's model contain a unit root. However, by construction of the latter model, the unit root appears in the trend component. Given the stationary representation of the AR process, the shocks to the cyclical component are transitory.

4.1.2 Friedman's plucking model

Kim and Nelson (1999*a*) proposed a model where the asymmetric behaviour is attached to the transitory component. We describe this model for completeness as it is a particular case of the general state space model given in Appendix (A.1). In our empirical analysis, we selected other models described in this section which showed a sensible performance for our data set. Kim and Nelson (1999*a*) motivate the model by relating it to Milton Friedman's economic model characterized by the observation that: the amplitude of a recession is strongly correlated with the subsequent expansion whereas the amplitude of an expansion is not correlated with the next recession. The model is defined below:

$$y_t = n_t + x_t + u_t \,, \qquad u_t \sim NID(0, \sigma_u^2) \,, \tag{16}$$

$$n_t = g_{t-1} + n_{t-1} + v_t \,, \qquad v_t \sim NID(0, \sigma_v^2) \,, \tag{17}$$

$$g_t = g_{t-1} + w_t \,, \qquad w_t \sim NID(0, \sigma_w^2) \,, \tag{18}$$

$$x_t = \sum_{i=1}^{k} \phi_i x_{t-i} + \delta_0 + \delta_1 S_t + e_t \,, \qquad e_t \sim NID(0, \sigma_e^2) \,, \tag{19}$$

where S_t is an indicator variable for the regimes modelled as a first order Markov process.

The model is intended to mimic the hypothesis that recessions are driven by transitory shocks while positive shocks are permanent. Kim and Nelson (1999*a*) also point out the fact that for $\sigma_\kappa^2 = 0$ and $\delta_0 + \delta_1 S_t < 0$ there is an upper limit to the trend, which agrees with the maximum feasible output discussed in Friedman's model.

4.1.3 Markov switching variances

Originally, the interest to include a Markov switching variable for business cycle analysis relied on the fact that it provides a measure of changes in the components during recession and expansion periods. McConnell and Pérez-Quirós (2000) have investigated the question whether the volatility of the business cycle has remained constant throughout time. The findings in the mentioned paper suggest a natural extension in our structural model by allowing for Markov switching variances in some of the disturbance terms. Changes in regime in the variance of the components can be specified as:

$$\sigma^2 = \sigma_1^2 S_{jt} + \sigma_2^2 S_{jt}\,, \tag{20}$$

where S_{jt} takes the value 1 when the j-th regime is governing the series and 0 otherwise. We consider two regimes, $j = 1, 2$. We do not address the question about changes in volatility in the business cycle, however, our empirical results in Section 5 show statistical evidence for structural breaks in the variance of GDP series using a model with switching variance. Some models performed better when some of the variances were allowed to change with regime. As we will see, we will also consider the possibility of a regime dependent covariance between the trend and the cycle components.

4.1.4 Switching damping factor

In applied works, a large estimate for the persistence of shocks in the transitory cycle is often obtained. In the AR representation, the sum of the AR coefficients is taken as a measure of persistence. In the trigonometric representation, the damping factor can be taken as such measure. We estimate the persistence parameter in the context of Markov switching models. The question that we address is whether there is empirical evidence for a lower persistence parameter in one regime compared to the other regime. We take the trigonometric representation of the cycle since it requires only one additional switching parameter, ρ_{S_t}, instead of k switching AR coefficients. We specify the following model:

$$
\begin{aligned}
y_t &= n_t + x_t + u_t\,, & u_t &\sim NID(0, \sigma_u^2)\,, & (21)\\
n_t &= n_{t-1} + \tau_0 + \tau_1 S_t\,, & & & (22)\\
x_t &= \rho_{S_t}\, x_{t-1}\cos\lambda + \rho_{S_t}\, x^*_{t-1}\sin\lambda + e_t\,, & e_t &\sim NID(0, \sigma^2_{e,S_t})\,, & (23)\\
x^*_t &= -\rho_{S_t}\, x_{t-1}\sin\lambda + \rho_{S_t}\, x^*_{t-1}\cos\lambda + e^*_t\,, & e^*_t &\sim NID(0, \sigma^{2*}_{e,S_t})\,. & (24)
\end{aligned}
$$

The damping factor varies from one regime to the other: $\rho_{S_t} = \rho_1 S_{jt} + \rho_2 S_{jt}$, where S_{jt} is an indicator variable that takes the value 1 when the j-th regime is governing the series and 0 otherwise. We consider two regimes, $j = 1, 2$.

4.1.5 Two transitory components

Motivated by the findings obtained while analyzing the GDP series for France, we considered the presence of two different transitory components. We selected the model defined below for the GDP series of France. It bears resemblance to Lam's model in that there is switching variable in the trend component. However, no difference is taken to the data, a model for the trend is given (a deterministic trend with switching) and two transitory components in the trigonometric representation are estimated:

$$
\begin{aligned}
y_t &= n_t + x_t + u_t\,, & u_t &\sim NID(0, \sigma_u^2)\,, & (25)\\
n_t &= n_{t-1} + \tau_0 + \tau_1 S_t\,, & & & (26)\\
x^{(1)}_t &= \rho^{(1)} x^{(1)}_{t-1}\cos\lambda^{(1)} + \rho^{(1)} x^{(1)*}_{t-1}\sin\lambda^{(1)} + e^{(1)}_t\,, & e_t &\sim NID(0, \sigma_e^{2\,(1)})\,, & (27)\\
x^{(1)*}_t &= -\rho^{(1)} x^{(1)}_{t-1}\sin\lambda^{(1)} + \rho^{(1)} x^{(1)*}_{t-1}\cos\lambda^{(1)} + e^{(1)*}_t\,, & e^{(1)*}_t &\sim NID(0, \sigma_e^{2\,(1)*})\,, & (28)\\
x^{(2)}_t &= \rho^{(2)} x^{(1)}_{t-1}\cos\lambda^{(2)} + \rho^{(2)} x^{(2)*}_{t-1}\sin\lambda^{(2)} + e^{(2)}_t\,, & e_t &\sim NID(0, \sigma_e^{2\,(2)})\,, & (29)\\
x^{(2)*}_t &= -\rho^{(2)} x^{(1)}_{t-1}\sin\lambda^{(2)} + \rho^{(2)} x^{(2)*}_{t-1}\cos\lambda^{(2)} + e^{(2)*}_t\,, & e^{(2)*}_t &\sim NID(0, \sigma_e^{2\,(2)*})\,. & (30)
\end{aligned}
$$

4.1.6 Correlated components

In subsection 2.2 we discussed the identification of the parameters in Clark's model. We explained that the conventional assumption for identification is to consider mutually uncorrelated components. We also discussed that, as Morley et al. (2003) show, the model with a random walk with deterministic drift for the trend component plus a transitory component is an ARIMA(2,1,2) model and is exactly identified. An identifying assumption $\sigma_{we} = 0$ commonly considered in practical applications is not necessary in this model.

Sinclair (2007) estimates the covariance between the components in the permanent plus transitory model with asymmetries in the transitory component. We will estimate the covariance between the trend and cycle components in two models. In both models the transitory component is estimated in the trigonometric representation. The first model considers Markov switching variances in both components as well as switching covariance:

$$y_t = n_t + x_t \tag{31}$$

$$n_t = n_{t-1} + v_t\,, \qquad v_t \sim NID(0, \sigma^2_{v,S_t})\,, \tag{32}$$

$$x_t = \rho x_{t-1}\cos\lambda + \rho x^*_{t-1}\sin\lambda + e_t\,, \qquad e_t \sim NID(0, \sigma^2_{e,S_t})\,, \tag{33}$$

$$x^*_t = -\rho x_{t-1}\sin\lambda + \rho x^*_{t-1}\cos\lambda + e^*_t\,, \qquad e^*_t \sim NID(0, \sigma^{2*}_{e,S_t})\,, \tag{34}$$

with $\sigma_{ev,S_t} \neq 0$. The switching variances were defined in subsection 4.1.3, the switching covariance is defined similarly. In the second model the variances and the covariance are not regime-dependent, a switching variable in the trend component, τ_{S_t}, is included instead:

$$y_t = n_t + x_t \tag{35}$$

$$n_t = n_{t-1} + \tau_0 + \tau_1 S_t + v_t\,, \qquad v_t \sim NID(0, \sigma^2_v)\,, \tag{36}$$

$$x_t = \rho x_{t-1}\cos\lambda + \rho x^*_{t-1}\sin\lambda + e_t\,, \qquad e_t \sim NID(0, \sigma^2_e)\,, \tag{37}$$

$$x^*_t = -\rho x_{t-1}\sin\lambda + \rho x^*_{t-1}\cos\lambda + e^*_t\,, \qquad e^*_t \sim NID(0, \sigma^{2*}_e)\,, \tag{38}$$

with $\sigma_{ev} \neq 0$.

4.2 Maximum Likelihood Estimation

The linear model underlying Clark's model can be estimated by maximum likelihood (ML) using the Kalman filter. For the extended non-linear model with Markov switching regimes, other algorithms have been developed. Lam (1990) developed an algorithm for exact ML when asymmetries are modelled in the trend component. Kim (1994) develops an algorithm for the estimation of a general model with Markov switching regimes given in a state space representation by approximate ML. This method is attractive for our general model, since it allows to estimate different versions of it.

4.2.1 Maximum likelihood and Kim's filtering algorithm

Kim's filtering algorithm is built upon the Kalman and Hamilton filters. Here, we sketch the implementation of the algorithm:

1. Run the Kalman filter for all the possible paths of the Markov process in the period t and $t-1$, $S_t = j, S_{t-1} = i$ with $i, j = 1, 2, ..., M$ and M the number of regimes. There are M^2 paths to consider leading to M^2 state values and variances.

2. Run the Hamilton filter and compute the weighting terms $Pr(S_t, S_{t-1}|\psi_{t-1})$. The variable ψ_{t-1} denotes the set of information available up to time $t-1$.

3. Collapse the resulting M^2 state values and the corresponding variance covariance matrix (for each path $S_t = j, S_{t-1} = i$ with $i, j = 1, 2, ..., M$) into M-vectors according to the following approximations:

$$\alpha_{t|t}^{(j)} = \frac{\sum_{i=1}^{M} Pr\left(S_t = j, S_{t-1} = i|\psi_t\right) \alpha_{t|t}^{(i,j)}}{Pr\left(S_t = j|\psi_t\right)},$$

$$P_{t|t}^{(j)} = \frac{\sum_{i=1}^{M} Pr\left(S_t = j, S_{t-1} = i|\psi_t\right) \left(P_{t|t}^{(i,j)} + \left(\alpha_{t|t}^{(j)} - \alpha_{t|t}^{(i,j)}\right)\left(\alpha_{t|t}^{(j)} - \alpha_{t|t}^{(i,j)}\right)'\right)}{Pr\left(S_t = j|\psi_t\right)}.$$

The Kalman and Hamilton filters are initialized and the above procedure is run for $t = 1, 2, ..., n$. At each iteration the contribution of each observation to the likelihood is obtained through the marginal density of the observation given past information , $f(y_t|\psi_{t-1})$. The log-likelihood function given the whole sample data is obtained in the last iteration as:

$$\text{log-likelihood} = \log(f(y_1, y_2, ..., y_n)) = \sum_{t=1}^{n} \log\left(f(y_t|\psi_{t-1})\right)$$

An optimization algorithm is then used to find the parameter values that maximize the previous log-likelihood function.

4.2.2 Exact and approximate maximum likelihood

Previous to the application of Kim's algorithm for the empirical analysis, we assess the performance of the procedure comparing results with two reference empirical applications for which procedures for exact ML exist. We estimate Clark's and Lam's model for the US GDP series used in the reference papers. Table 1 summarizes the comparison between exact and approximate ML.

[Table 1 about here.]

Parameter estimates in Clark's model are almost identical for both methods. This is not surprising, since the model can be regarded as a single regime model where the probability of leaving regime 1 is fixed to 0. The approximations involved in Kim's algorithm lead to slight differences with the results in Lam's model. The log-likelihood is larger when exact ML is carried out, however, there are no substantial differences in the implied cycle. Figure 1 shows that the implied cyclical component and the filtered probabilities of state 1 from both procedures are close to each other.

[Figure 1 about here.]

The difference at the beginning of the sample is presumably due to the fact that in the original application considered for comparison the initial observations are discarded for computation of the likelihood function whereas we estimate the initial values of the state vector. The number of observations used in both cases are the same, nonetheless, and the value of the log-likelihood at the optimal parameters are comparable. Finally, the variance in the observation equation, σ_u^2, is also estimated despite the original model only includes a variance in the trend component, σ_v^2. While the original application fits the model to the first difference of the data, in the state space representation that we follow the explanatory variable is the original data. In order to account for a larger variance in the non-differenced data, a disturbance term is estimated in the observation equation and the trend component.

We do not claim that the choice of exact or approximate ML is irrelevant, even if results are reasonably close. The previous exercises are intended to illustrate that, following the framework introduced above, our implementation allows us to obtain results in close agreement with empirical results found in the literature. In what follow, we will take the same approach for extended versions of Clark's and Lam's models for which exact ML is not feasible.

5 Empirical Results

In this section the methods described above are applied to a data set consisting of quarterly gross domestic product (GDP) time series. GDP is defined as the aggregation of the value of all goods and services produced less the value of any goods or services used in their creation. It is commonly interpreted as a measure of economic activity and is a reference for analyzing the business cycle.

[Table 2 about here.]

The data set is described in Table 2. The GDP series in the data are measured in the national currency at constant prices. Unless otherwise stated, the data are transformed into the logarithmic scale. The series are obtained from the national accounts: the ESA95 data base for the European countries is available at `http://sdw.ecb.europa.eu` and the US Economic Accounts data base is available at `http://research.stlouisfed.org/fred2`. The series related to the European countries are working day and seasonally adjusted and the US GDP is seasonally adjusted annual rate. Seasonally adjustment procedures extract from the series those fluctuations with periodicity lower than a year that are regularly observed in the data.

We will illustrate and apply the methods discussed throughout the thesis for the GDP data set. Firstly, we illustrate the contribution from dynamic econometric models to the empirical analysis separately for the unobserved components model and Markov switching regime models. Secondly, we explore further empirical issues in the context of the Clark model with Markov switching regimes; namely, the presence of switching damping factor in the cyclical component, contemporaneous correlation between the innovation terms and a model with two transitory components.

The required computations were implemented in the `R` statistical language R Development Core Team (2006).

5.1 Structural Time Series Models

Table 3 shows maximum likelihood parameter estimates in the trend plus cycle structural model with the trigonometric representation for the cycle, equations (1)-(2) and (5)-(6). The variance of the disturbance term u_t in the observation equation is fixed to 0. The inferred filtered cycles are shown in Figure 2.

[Table 3 about here.]

[Figure 2 about here.]

In the series FR and UK, the value zero is included within a two-standard-error interval around the estimated standard deviations for the level and the slope, σ_v and σ_w. The trend is closer to a deterministic pattern in these series. The damping factor ρ is estimated to be close to unity in all cases except NL. The transitory component in the series NL is not reliably estimated, as Figure 2 also suggests. The estimated frequency in the series UK is the lowest and entails a cycle with a period too long (81 quarters) for it to be interpreted as a business cycle. In the series FI, FR and ES a cycle around 10 years long is identified. A shorter cycle, 5 years long, is identified for US.

[Table 4 about here.]

[Figure 3 about here.]

Table 4 reports parameter estimates in the trend plus cycle structural model with a stationary AR(2) process for the cycle, equations (1)-(4). The variance of the disturbance term u_t in the observation equation is fixed to 0. The inferred filtered cycles are shown in Figure 3. The estimated cycle in the series FR remains similar to that obtained using the trigonometric representation of the cycle. The estimation of the cycle in the series NL does not improve: both estimates are very similar, see Figures 2 and 3, and different phases of the cycle cannot be reliably identified. The cycle estimated for US replicates the results obtained in other works using the same specification, see for instance Kim and Nelson (1999*b*, Chapter 3), and are updated for a longer sample period. The estimate of the cycle for UK is noisy and not satisfactory using this specification. Looking at the major episodes: expansion in the 1960s, recession in the beginning of the 1980s and 1990s and subsequent revivals, the cycle estimated for UK with the trigonometric specification resembles the cycle for US with the AR(2) specification in the cycle component.

5.2 Markov Switching Models

We fit an autoregressive model with Markov switching in mean (AR-MS) to the data set of GDP series. Estimates were obtained by maximum likelihood using the Hamilton filter. Some observations (indicated in Figure 5) were found to be significant additive outliers and dummy variables were included accordingly. No other type of outliers (level change, temporal change or innovative outliers) were considered. The order of the AR model was chosen according to the significance of lagged variables at the 5% level. All the roots of the fitted autoregressive polynomial lie outside the unit circle.

We consider the existence of two regimes. Testing for the null hypothesis of no regime against the alternative of two regimes is not straightforward. That issue is an instance of a broader subject studied in the econometric literature that involves inference when a nuisance parameter is not identified under the null hypothesis. In our context, under the null $\mu_1 = \mu_2$ in model (11), the transition probabilities p_{ii} $i = 1, 2$ are not identified. As a consequence, the test statistics based on maximum likelihood do not follow the standard distributions. Some approaches have been studied, see García (1998) and references therein for an exposition of the problem. The proposed solutions found in the literature depend on the specification of the model and none of them appear to be fully satisfactory for practical purposes. We will focus on the estimation of a MS model and will check the performance of the model considering the existence of two regimes.

[Table 5 about here.]

Table 5 reports parameter estimates in the AR-MS model. In all cases except FR the mean is estimated to be negative in the first regime and positive in the second regime. The estimated switching means are separated by more than two standard deviations in all series except US. This suggests the interpretation of regime 1 as a recession period and regime 2 as a period of economic expansion. This fact is not sufficient to justify this interpretation since recession and expansion periods involve further economic considerations, among them, the duration of the phases of the business cycle.

The expected duration of each regime $i = 1, 2$ can be obtained from Table 5 as $\frac{1}{1-p_{ii}}$. The expected durations of regime 1 are: 12, 3, 1, 2, 2 and 3 quarters, respectively for the six series as ordered in Table 5. Likewise, the expected durations of regime 2 are: 111, 8, 21, 10, 16 and 17 quarters. These durations agree with the general understanding that the business cycle alternates short recessionary periods and longer expansion periods.

[Figure 4 about here.]

[Figure 5 about here.]

The graphical analysis of results (Figures 4 and 5) reveals that the link between the fitted regimes and the interpretation of recession and expansion periods is not clear in some cases. The results for the series FR, UK and US are closer to the understanding of recession and expansion periods. In the series FI, high probabilities of regime 1 during the period 1990-1994 appear to be related to a break in the series in that period. Identification of two regimes in the series NL is not plausible; the probability p_{11} is nearly zero and the few observations likely to belong with regime 1 are related to a few downward peaks in the mean, which remains relatively constant especially at the end of the sample.

Although not reported, according to the Jarque-Bera test statistic, we found that the hypothesis of normally distributed residuals cannot be rejected at the 5% significance level in all series except ES. Considering an AR model without switching mean, the Jarque-Bera test statistic was far beyond the 5% critical value for the null of normality. An analysis of the series with the program TRAMO –Gómez and Maravall (1996)– detected different outliers other than additive outliers (level shifts and temporary changes).

Taking logarithms is a common practice in time series analysis, since it homogenizes the variance and abates the effect of potential outliers. Indeed, that practice is often recommended in macroeconomic time series by the statistical analysis, for instance by means of the range-mean plot. In the series UK, however, to take logarithms was not supported by that analysis since the pattern of heteroscedasticity exhibits a decrease in the variance of the series, which is accentuated in the logarithmic scale. A further insight into this issue applying the Golfeld-Quandt (GQ) test statistic for homoscedasticity for different break points raised the question whether a structural change is present in the series. To investigate this issue, we fit to the series UK an AR model with a two-state Markov switching variance. We find that the analysis based on the AR-MS model suggests a structural change in the data.

[Figure 6 about here.]

Figure 6 displays the growth rates of the UK GDP and the filtered probabilities of regime 1 when an AR model with a Markov switching both in mean and variance is fitted. We can see that the highest probabilities of regimes cluster in two periods: the first three quarters of the sample size and the last quarter of observations, approximately. The estimated variances are 1.339 and 0.068, respectively for regime 1 and 2. Another possibility we considered is to capture this effect by means of a dummy variable for the last years of the sample. Alternatively, we considered a dummy in the switching mean and variance. In this case, different average growth rates and variances in each regime can be estimated for each period determined in the dummy variable. The switching mean and variance are specified as follows:

$$\mu_{S_t} = (\mu_1 + \mu_1^* D_t) S_{1t} + (\mu_2 + \mu_2^* D_t) S_{2t} \,,$$
$$\sigma^2_{S_t} = (\sigma_1^2 + \sigma_1^{2^*} D_t) S_{1t} + (\sigma_2^2 + \sigma_2^{2^*} D_t) S_{2t} \,,$$

where D_t is a dummy variable consisting of zeros in the beginning of the sample and ones for the most recent years. Neither of the approaches provided conclusive results. In the first case the dummy was not significant at the 5% level. The large number of parameters involved in the second case made the optimization procedure relatively more sensitive to the starting values.

A less burdensome approach compared to the selection of dummy variables is to apply the Box-Cox transformation. It is defined as follows:

$$y_{bc} = \frac{y^\kappa - 1}{\kappa} \,, \quad \kappa \neq 0 \,,$$

where $\lim_{\kappa \to 0} y_{bc} = \log(y)$ and, hence, setting a parameter $\kappa = 0$ is equivalent to take logarithms. We use the Box-Cox transformation with $\kappa = 1$ for the series UK. Similar results were observed for the series US and we set $\kappa = 0.5$ to homogeneize the variance of the series. In the other series we follow a logarithmic transformation. Despite this transformation accounts for an evolving variance rather than a break, the performance was better than in the dummy approach discussed above and it is computationally less burdensome. Furthermore, the GQ test did not reject the null of homoscedastic residuals in the series US. The disadvantage of this approach is that the resulting transformed data do not have a direct interpretation in terms of growth rates.

The series NL is an interesting case illustrating the ability to capture certain types of non-linearities by means of the AR-MS model. The program TRAMO indicated the presence of significant outliers, most of them level shifts and temporal changes. Omitting those regressor variables leads to the rejection of the null hypothesis of normally distributed residuals. Interestingly enough, the Jarque-

Bera test statistic for normality turned out to be well below the critical value for the 5% significance level in a model with a switching mean and one additive outlier for the observation 1979:I.

Leaving out possible interpretations of the regimes, Markov switching models may be a useful tool for detecting and/or modelling structural breaks and certain type of outliers such as level shifts and temporal changes broadly found in macroeconomic time series. This conclusion is especially relevant in the estimation procedure that we followed, where we did not include any explicit constraints in the parameters (for instance the mean in regime 1 to be lower or greater than in regime 2), therefore, there is no *a priori* interpretation of the parameters. The parameters account for any kind of non-linearities likely to be captured by the Marvov process. We found that, *a posteriori*, non-linearities detected in some of the series are in agreement with the understanding of the phases of the business cycle, whereas in other cases the non-linear model reveals the presence of a structural change or outlier observations.

5.3 Structural Model with Markov Switching Regimes

In what follows, we explore further empirical questions by means of extensions of the standard reference models. In particular, we estimate a switching damping factor in the cyclical component for the series FR and US; contemporaneous correlation between the innovation terms is estimated in the US GDP series and two transitory components are detected in the GDP series for France.

5.3.1 Switching damping factor

The degree of persistence in macroeconomic time series is often sensitive to the inclusion of intervention variables and the sample period considered. Here, we address this issue by means of a regime-dependent persistence parameter. A switching variable in the trend and a switching variance in the transitory component are also considered. We use the trigonometric specification of the transitory component and consider the damping factor ρ as a measure of persistence. In the AR(2) specification, the sum of the two AR coefficients is commonly interpreted as a measure of persistence of shocks. Considering changes in regime using this measure of persistence would entail to estimate two switching parameters while in the trigonometric specification it simplifies to one switching parameter.

[Table 6 about here.]

Table 6 summarizes the results for the US and FR GDP series. We find that, for the selected model, a switching damping factor is not plausible in the US GDP series. The parameter estimates take a value larger than 0.9, being close to unity in both regimes. In the GDP series for France, the damping factor is substantially lower in regime 1 than in regime 2. The difference in the parameters is larger than two times the standard errors.[1]

5.3.2 Correlated components

We analyze the presence of contemporaneous correlation between the trend and cycle components in the context of Markov switching models. Table 7 summarizes the results for two models applied to the US GDP series.

[Table 7 about here.]

The first model, defined in equations (31)-(34), includes two-state Markov switching variances and covariance. The average durations of each regime implied by the estimated probabilities were not expected, since they are too short. Nevertheless, we can see that two variances and two covariances are estimated for each regime. The covariance in regime 1 is relatively different from zero compared to the estimate for regime 2.

We consider a second model, equations (35)-(38), without switch in the variances and covariance but a switching variable in the trend component. As expected, this lead to a lower variance in the trend component. The covariance turns to be negative in this case, although the identification of this parameter is not fully satisfactory as the relatively large standard error suggests.

[Figure 7 about here.]

The implied cycle and filtered probabilities of regime 2 for model 2 are shown in Figure 7. Official trough dates reasonably match with our results in most cases except for the recession at the beginning of 1970, where the second phase of the cycle is stretched compared to the official business cycle dates.

5.3.3 Two transitory components

Here, we explore the plausibility of two transitory components in the GDP of France. In the analysis of this series, we found that, depending on the model, in some cases a cycle of longer period and of deterministic nature was detected, whereas in other cases a stochastic cycle of lower periodicity was detected. Top-left plot in Figure 8 displays the spectral density estimate for the FR GDP series after removing a linear trend to render stationarity. Two humps are observed. The first one is prominent, while the second is in the boundary of the 95% confidence bar. We have a further insight into this

[1] The estimator for the parameter ρ is not normally distributed. As a rule of thumb, we take a two-standard-error interval.

issue by jointly estimating two potential transitory components. We specify a model that consists of a deterministic linear trend with Markov switching and two cyclical components defined in the trigonometric form. Results are shown in Table 8.

[Table 8 about here.]

The switching parameter related to regime 1 is not well estimated, as the large standard error indicates. Despite this circumstance, the performance of the model was relatively better than other tentative models used in the analysis. One of the cycles is purely deterministic (the corresponding standard deviation tends to zero) with periodicity $2\pi/0.15 \approx 42$ quarters. The standard deviation related to the second cycle is different from zero and the periodicity is $2\pi/0.298 \approx 21$ quarters.

[Figure 8 about here.]

The inferred cycles (Figure 8) resemble those that were individually found in other single cycle models. Filtered probabilities of regime 1 are also shown in this figure. One cycle of the component with lower periodicity is completed within half cycle of the other component, approximately. The highest probabilities of regime 1 closely coincide with a trough in the cycle of higher periodicity, especially at the beginning of the sample.

6 Conclusions

In this thesis we set up a framework for business cycle analysis by means of a structural time series model. We adopt Clark's permanent plus transitory unobserved components model and extend it with Markov switching regimes in the components and parameters of the model. Our reference framework encompasses preliminary extensions of Clark's model found in the literature accounting for asymmetries in the components.

We discuss approximate maximum likelihood estimation in our non-linear structural time series model using Kim's filtering algorithm. We conduct two applications for which exact ML methods exist and find that results based on our implementation of the approximate ML procedure are close to those based on exact ML. Then, we go further into the analysis of other versions of the reference structural model for which exact ML methods do not exist.

We apply Clark's model to quarterly gross domestic product time series. A cycle around 10 years long is identified in most of the series. A shorter cycle, 5 years long, is identified for US. The analysis of the same data in an autoregressive model with switching variance suggests the presence of a structural break in the GDP series of UK. As a complement to the common practice of considering a switching mean, a switching variance in GDP series is found to be relevant. We also find that non-linearities detected in some of the series are in agreement with the understanding of the phases of the business cycle, whereas in other cases the non-linear model reveals the presence of a structural change or outlier observations.

The benchmark model discussed in this thesis provides a unified framework for the analysis of the business cycle. It makes it possible the selection of different specifications for the study of particular questions within the same framework. The model is illustrated in the joint context of the unobserved components model and Markov switching regimes for the GDP series of France and USA.

We investigate empirical questions that have been addressed in the literature separately for structural and Markov switching models. A model with switching damping factor is estimated for the GDP of France and USA. Results for the GDP of France suggest the presence of lower persistence of shocks in the regime where a lower variance is attached to the cyclical component. Contemporaneous correlation between components is estimated in two different models. The sign of the correlation varies when correlation is allowed to change between regimes and when it is fixed in a model with a switch in the trend. Finally, the presence of two transitory components is discussed in a model with asymmetries in the trend for the GDP of France. A deterministic cycle with periodicity 42 quarters and a stochastic cycle with periodicity 21 quarters are detected.

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A Appendix

A.1 State Space Representation

A state space representation for the model with a permanent component and a transitory component in AR form with Markov switching regimes is given by:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} n_t \\ g_t \\ x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix} + u_t \,,$$

$$\begin{bmatrix} n_t \\ g_t \\ x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix} = \begin{bmatrix} \tau_{S_t} \\ 0 \\ \delta_{S_t} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \phi_1 & \phi_2 & \cdots & \phi_p \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_{t-1} \\ g_{t-1} \\ x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p} \end{bmatrix} + \begin{bmatrix} v_t \\ w_t \\ e_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

For the trigonometric representation of the transitory component, equations (5)-(6), we follow the state space representation:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_t \\ g_t \\ x_t \\ x_t^* \end{bmatrix} + u_t \,,$$

$$\begin{bmatrix} n_t \\ g_t \\ x_t \\ x_t^* \end{bmatrix} = \begin{bmatrix} \tau_{S_t} \\ 0 \\ \delta_{S_t} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho\cos\lambda & \rho\sin\lambda \\ 0 & 0 & -\rho\sin\lambda & \rho\cos\lambda \end{bmatrix} \begin{bmatrix} n_{t-1} \\ g_{t-1} \\ x_{t-1} \\ x_{t-1}^* \end{bmatrix} + \begin{bmatrix} v_t \\ w_t \\ e_t \\ e_t^* \end{bmatrix} .$$

A.2 The Hamilton Filter

The weighting terms $Pr\left(S_t, S_{t-1}, ...S_{t-p}|\psi_{t-1}\right)$ are the probability of each one of the possible paths. They are computed in a two-step prediction-update stage:

1. Given $Pr\left(S_{t-1}=i,...,S_{t-p}=k\right)$ from the previous iteration, at the beginning of the t-th iteration calculate the probability of each path:[2]

$$\begin{aligned} & Pr\left(S_t=j,S_{t-1}=i,...,S_{t-p}=k|\psi_{t-1}\right) \\ & =Pr\left(S_t=j|S_{t-1}=i,...,S_{t-p}=k,\psi_{t-1}\right)\times Pr\left(S_{t-1}=i,...,S_{t-p}=k|\psi_{t-1}\right)\end{aligned}$$

by the Markov property

$$=Pr\left(S_t=j|S_{t-1}=i\right)\times Pr\left(S_{t-1}=i,...,S_{t-p}=k|\psi_{t-1}\right)\ , \tag{39}$$

for all paths with $i,j,k=1,2$. The Markov property of the first order Markov process S_t entails that, conditional on the the process one period backward, S_{t-1}, the present S_t and S_{t-i} for $i>1$ are independent and, hence, at time $t-1$ all the information about S_t is contained in S_{t-1}. Thus, the first term is the transition probability of the Markov process.

2. For the update step it is convenient to write the updated set of information ψ_t as (ψ_{t-1},y_t). At the end of the t-th iteration, the previous joint probability is updated using one additional observation y_t:

$$\begin{aligned} & Pr\left(S_t=j,S_{t-1}=i,...,S_{t-p}=k|\psi_{t-1},y_t\right) \\ & =\frac{f\left(y_t,S_t=j,S_{t-1}=i,...,S_{t-p}=k|\psi_{t-1}\right)}{f\left(y_t|\psi_{t-1}\right)} \\ & =\frac{f(y_t|S_t=j,S_{t-1}=i,...,S_{t-p}=k,\psi_{t-1})Pr(S_t=j,S_{t-1}=i,...,S_{t-p}=k|\psi_{t-1})}{f(y_t|\psi_{t-1})}\ .\end{aligned} \tag{40}$$

The denominator of the previous expression is the marginal density computed according to equation (12). As the denominator is the sum for all possible paths of the term in the numerator, it has the effect of a proportionality factor that ensures that the resulting ratio is actually a probability taking a value in the $[0,1]$ interval.

It is important notice that although the second term in equation (39) can be decomposed into the product of transition probabilities, there is no need to go further in the decomposition. At this step, we can compute a new value of this term integrating S_{t-p} out of the expression in equation (40):

$$Pr\left(S_t=j,...,S_{t-p+1}=k'|\psi_t\right)=\sum_{S_{t-p}=1}^{2}Pr\left(S_t=j,S_{t-1}=i,...,S_{t-p}=k|\psi_t\right)\ .$$

This value will be used in the next iteration as input for the step in equation (39).

[2] Recall that in general there are s^{p+1} paths (where s is the number of regimes or states) and each path consists of $p+1$ observations of the Markov process or state variable.

Initialization of the filter can be done using the steady state probabilities. For a two-state first order Markov process, those probabilities are given by:

$$Pr(S_0 = 1|\psi_0) = \frac{1 - p_{22}}{2 - p_{11} - p_{22}}\,, \quad Pr(S_0 = 2|\psi_0) = \frac{1 - p_{11}}{2 - p_{11} - p_{22}}\,.$$

The series of filtered probabilities are then obtained running the entire procedure –involving equations (12), (39) and (40)– iteratively from $t = p + 1, p + 2, ..., n$ (with p the AR lag order) for the optimal parameter estimates using $\hat{p}_{11}$ and $\hat{p}_{22}$ as reference to compute the steady state probabilities.

Table 1: Clark's and Lam's model by exact and approximate ML

Clark's model, equations (1)-(4), is fitted to the US GDP series in the sample period 1952:II-1995:III. Lam's model, equations (13)-(15), is fitted to the US GDP series in the sample period 1952:II-1984:IV. Results by exact maximum likelihood are taken from Kim and Nelson (1999*b*, Chapter 3) and Lam (1990). The models are fitted by approximate ML using Kim's filtering algorithm. Results from exact and approximate ML are close to each other.

	Clark's model		Lam's model	
	Exact	Approx.	Exact	Approx.
p_{11}	-	-	0.508	0.560
p_{22}	-	-	0.957	0.932
σ_u	-	-	-	0.274
σ_v	0.0056	0.0056	0.771	0.620
σ_w	0.0002	0.0002	-	-
σ_e	0.0061	0.0061	-	-
τ_1	-	-	-1.483	-0.953
τ_2	-	-	2.447	1.924
ϕ_1	1.5346	1.5344	1.244	1.391
ϕ_2	-0.5888	-0.5884	-0.382	-0.484
Log-Lik.	578.52	578.54	-174.97	-180.33

Table 2: Quarterly gross domestic product data set

The source data bases are the European system of national accounts ESA95 and US national accounts. The data for the European countries are measured in millions of euros at constant prices and for USA in billions of chained 2000 US dollars. The naming convention followed in the source data base for the European countries is 'ESA.Q.xx.Y.0000.B1QG00.1000.TTTT.Q.N.A' where 'xx' is the country code: FI, FR, NL, ES and GB. The identification code for the US series in the source data base is GDPC96.

Country	Span period	# obs.
Finland	1975:I-2007:IV	132
France	1978:I-2008:I	121
Netherlands	1977:I-2008:I	125
Spain	1980:I-2008:I	113
United Kingdom	1955:I-2008:I	213
USA	1947:I-2007:IV	244

Table 3: Parameter estimates in the trend plus cycle model with the trigonometric representation of the cycle

Cells report parameter estimates in the trend plus cycle structural time series model with the trigonometric representation for the cycle, equations (1)-(2) and (5)-(6). The variance of the disturbance term u_t in the observation equation is fixed to 0. The first 21 observations are discarded in the computation of the likelihood. Standard errors are reported in parentheses. The periodicity of the cycle of frequency λ is $2\pi/\lambda$ quarters. Figure 2 displays the inferred cyclical components.

Series	σ_v	σ_w	σ_e	ρ	λ	$2\pi/\lambda$	Log-Lik.
FI	0.0093 (0.0007)	0.0017 (0.0009)	0.0001 (0.0026)	0.9916 (0.0081)	0.1781 (0.0100)	35	353.41
FR	0.0001 (0.0002)	0.2199×10^{-6} (0.0002)	0.0033 (0.0001)	0.9720 (0.0161)	0.1549 (0.0188)	41	426.06
NL	0.0077 (0.0011)	0.0003 (0.0007)	0.0025 (0.0017)	0.0042 (0.0316)	0.2940 (−)	21	349.68
ES	0.0075 (0.0005)	0.1470×10^{-8} (0.0005)	0.2722×10^{-6} (0.0013)	0.9805 (0.0122)	0.1579 (0.0150)	40	323.14
UK	0.3060×10^{-6} (0.0016)	0.2815×10^{-8} (0.0004)	0.0091 (0.0004)	0.9557 (0.0172)	0.0776 (0.0261)	81	629.25
US	0.0001 (0.0029)	0.0012 (−)	0.0071 (0.0003)	0.8956 (0.0221)	0.3228 (0.0144)	19	736.16

Table 4: Parameter estimates in the model with trend plus a stationary AR(2) cyclical component.

Cells report parameter estimates in the trend plus cycle structural time series model with the autoregressive representation for the cycle, equations (1)-(4). The variance of the disturbance term u_t in the observation equation is fixed to 0. The first 21 observations are discarded in the computation of the likelihood. Standard errors are reported in parentheses. The roots refer to the polynomial $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$. Figure 3 displays the inferred cyclical components.

Series	σ_v	σ_w	σ_e	ϕ_1	ϕ_2	Roots $\phi(L)$	Log-Lik.
FR	0.0027 (0.0004)	0.1156×10^{-7} (0.0005)	0.0022 (0.0005)	1.6680 (0.0334)	-0.6955 (0.0044)	-0.4967; 2.8949	419.40
NL	0.0078 (0.0010)	0.0003 (0.0007)	0.0015 (0.0005)	-0.6564 (−)	-0.1077 (−)	1.2621; -7.3568	350.06
UK	0.0090 (0.0011)	0.3547×10^{-8} (0.0005)	0.0016 (0.0038)	-0.3226 (1.2453)	-0.0083 (0.0677)	2.8856; -41.7531	624.65
US	0.0055 (0.0019)	0.0001 (0.0005)	0.0061 (0.0020)	1.4798 (0.1698)	-0.5474 (0.1723)	-0.5598; 3.2632	743.65

Table 5: AR model with Markov switching in mean. Parameter estimates

Cells report parameter estimates for an AR model with a two-state Markov switching mean given in equation (11). Standard errors in parentheses. The dependent variable is 100 times the difference of the logarithms of the GDP series, except for UK and US where a Box-Cox transformation with Box-Cox parameter 1 and 0.5, respectively for each series is applied.

Series	p_{11}	p_{22}	ϕ_1	ϕ_2	ϕ_3	μ_1	μ_2	σ^2	Log-Lik.
FI	0.916 (0.075)	0.991 (0.009)	0.142 (0.068)	0.053 (0.062)	0.2335 (0.059)	-0.817 (0.265)	0.900 (0.098)	0.361 (0.046)	-124.398
FR	0.633 (0.165)	0.882 (0.129)	0.115 (0.132)	0.2534 (0.121)	0.219 (0.150)	0.105 (0.176)	0.628 (0.117)	0.104 (0.031)	-53.512
NL	1.891×10^{-5} (0.477)	0.952 (0.024)	0.021 (0.095)	0.172 (0.076)	-	-2.124 (0.467)	0.706 (0.079)	0.456 (0.061)	-145.325
ES	0.547 (0.116)	0.897 (0.035)	0.266 (0.049)	0.304 (0.039)	0.214 (0.041)	-0.205 (0.162)	0.861 (0.137)	0.087 (0.013)	-61.559
UK	0.452 (0.133)	0.939 (0.022)	0.040 (0.062)	0.241 (0.064)	0.247 (0.058)	-0.113 (0.024)	0.105 (0.0112)	0.006 (0.001)	163.414
US	0.676 (0.156)	0.939 (0.032)	0.164 (0.096)	0.150 (0.075)	-	-0.218 (0.241)	0.702 (0.082)	0.216 (0.027)	-191.803

Table 6: Switching damping factor

Cells report parameter estimates in a model with a Markov switching variable in the trend and switching variance in the transitory component, equations (21)-(24). Standard errors in parentheses. The damping factor ρ is taking as a measure of persistence and is estimated in two different regimes. The transitory component is represented in the trigonometric specification. The input series are 100 times the logarithm of the US GDP and FR GDP. A switching damping factor appears to be not plausible in the US GDP series. In the GDP series for France, the damping factor is substantially lower in regime 1 than in regime 2.

FR GDP				US GDP			
p_{11}	0.866 (0.113)	τ_1	-0.174 $(-)$	p_{11}	0.993 (0.009)	τ_1	-0.095 (818.643)
p_{22}	0.952 (0.039)	τ_2	0.004 (0.035)	p_{22}	0.995 (0.005)	τ_2	0.172 (0.0290)
ρ_1	0.662 (0.089)	σ_u^2	$0.8 \cdot 10^{-14}$ $(-)$	ρ_1	0.910 (0.027)	σ_u^2	$0.8 \cdot 10^{-12}$ (0.026)
ρ_2	0.999 (0.024)	$\sigma_{e,1}^2$	0.039 (0.014)	ρ_2	0.959 (0.020)	$\sigma_{e,1}^2$	0.191 (0.044)
λ	0.148 (0.017)	$\sigma_{e,2}^2$	0.120 (0.020)	λ	0.131 (0.022)	$\sigma_{e,2}^2$	1.219 (0.148)
Log-Lik.	-47.143				-271.265		

Table 7: Structural Markov switching model with correlated components

Cells report parameter estimates in two models with correlation between the trend and cycle components in the context of Markov switching models. Standard errors in parentheses. The input series is 100 times the logarithm of the US GDP. In both models the transitory component is estimated in the trigonometric representation. In model 1, equations (31)-(34), Markov switching variances are estimated in both components. The covariance is regime-dependent as well. The estimated covariance in regime 1 is relatively different from zero compared to the estimate for regime 2. In model 2, equations (35)-(38), the variances and the covariance are not regime-dependent, a switching variable in the trend component is included instead. The estimated covariance is negative in this model.

Parameter estimates			
Model 1		Model 2	
p_{11}	0.002	p_{11}	0.935 (0.030)
p_{22}	0.004	p_{22}	0.634 (0.120)
λ	0.138	λ	0.100 (0.022)
$\sigma^2_{v,1}$	0.050	τ_1	-0.232 $(-)$
$\sigma^2_{v,1}$	1.076	τ_2	1.789 (0.308)
$\sigma^2_{e,1}$	0.156	σ^2_v	$3.25 \cdot 10^{-9}$ $(-)$
$\sigma^2_{e,1}$	0.001	σ^2_e	1.454 (1.180)
$\sigma_{ev,1}$	0.361	σ_{ev}	-0.439 (0.569)
$\sigma_{ev,2}$	0.036	Log-Lik.	-332.695
Log-Lik.	-345.864		

Table 8: FR GDP series: Two transitory/cyclical components

Cells report parameter estimates in model (25)-(30) including a trend plus two transitory components in the trigonometric representation. Two transitory components are estimated jointly. The corresponding parameter estimates are labeled as (1) and (2), respectively for each transitory component. A Markov switching variable in the trend component is considered. Standard errors in parentheses. The input series is 100 times the logarithm of the GDP of France. Two cycles are detected. One of them is deterministic with periodicity 42 quarters. The second cycle is stochastic with periodicity 21 quarters.

Parameter estimates			
p_{11}	0.611 (0.129)	$\lambda^{(1)}$	0.298 (0.017)
p_{22}	0.913 (0.042)	$\lambda^{(2)}$	0.150 (0.003)
τ_1	-0.423 (222.768)	σ_u	0.169 (0.022)
τ_2	0.622 (0.075)	$\sigma_e^{(1)}$	0.110 (0.030)
Log-Lik.	45.406	$\sigma_e^{(2)}$	0.950×10^{-6} (0.0188)

Figure 1: Lam's model by exact and approximate ML

Based on results in Table 1. Lam's model, equations (13)-(15), is fitted to the US GDP series in the sample period 1952:II-1984:IV. The reference cyclical component and probabilities are computed following parameter estimates reported in Lam (1990). The cyclical component and probabilities estimated by approximate maximum likelihood are obtained using Kim's algorithm.

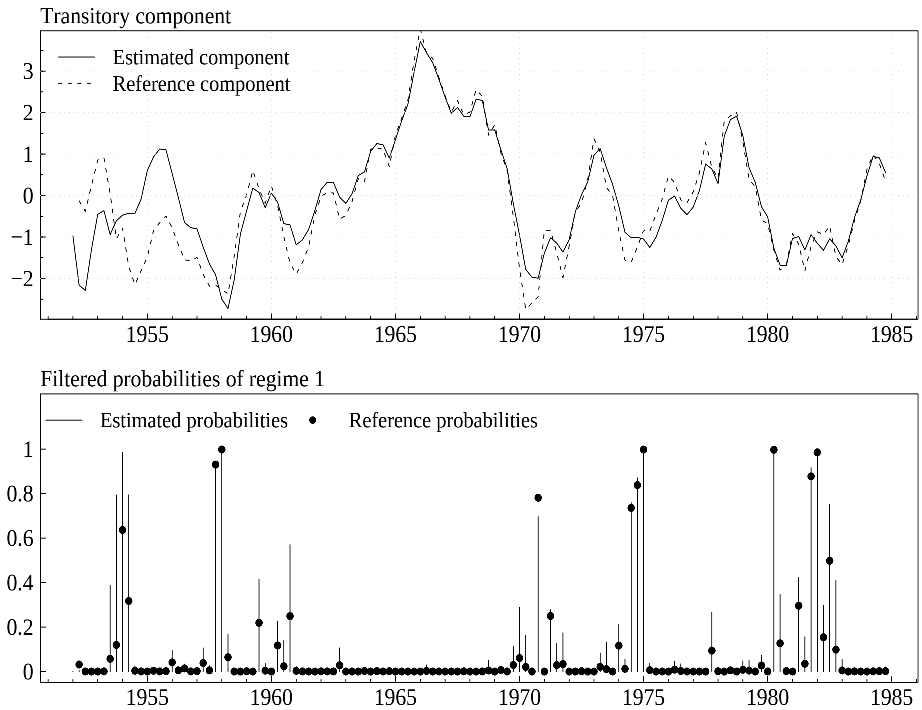

Figure 2: Cyclical component in trend plus cycle model with the trigonometric representation of the cycle

Inferred cyclical component from the permanent plus transitory unobserved components model with the trigonometric representation for the cycle, equations (1)-(2) and (5)-(6). Based on results in Table 3.

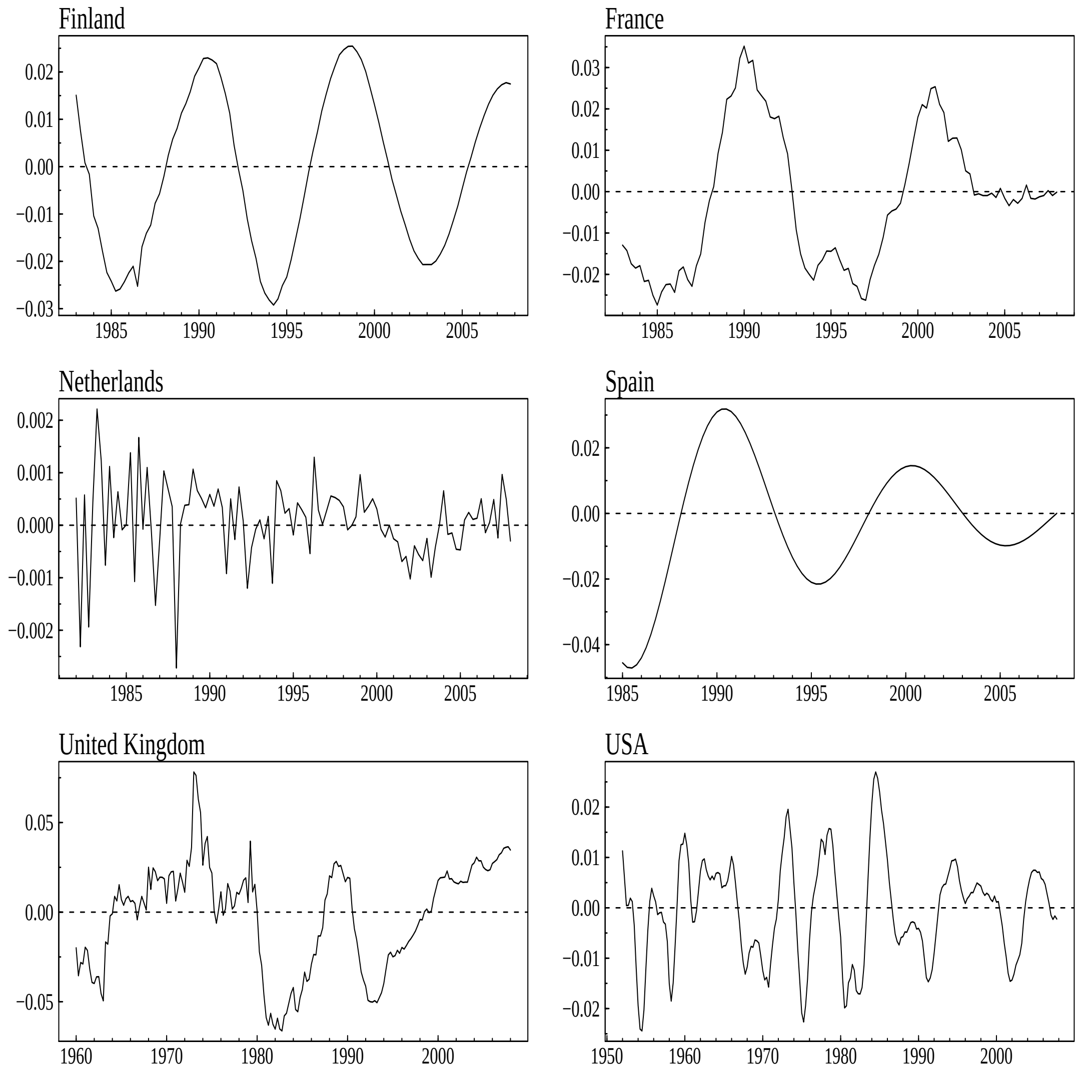

Figure 3: Cyclical component in the model with trend plus a stationary AR(2) cyclical component

Inferred cyclical component from the permanent plus transitory unobserved components model with an AR(2) model for the cycle, equations (1)-(4). Based on results in Table 4.

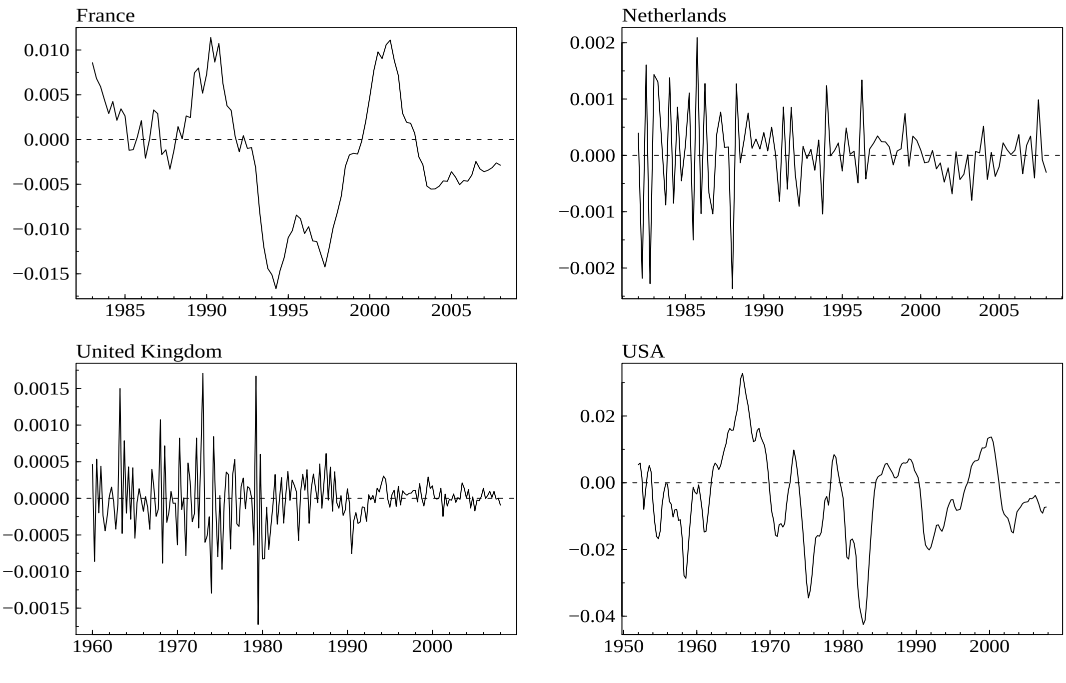

Figure 4: MS AR model. Filtered probabilities of regime 1

Filtered probabilities of regime 1 in an AR model with a two-state Markov switching in the mean given in equation (11). Based on results in Table 5.

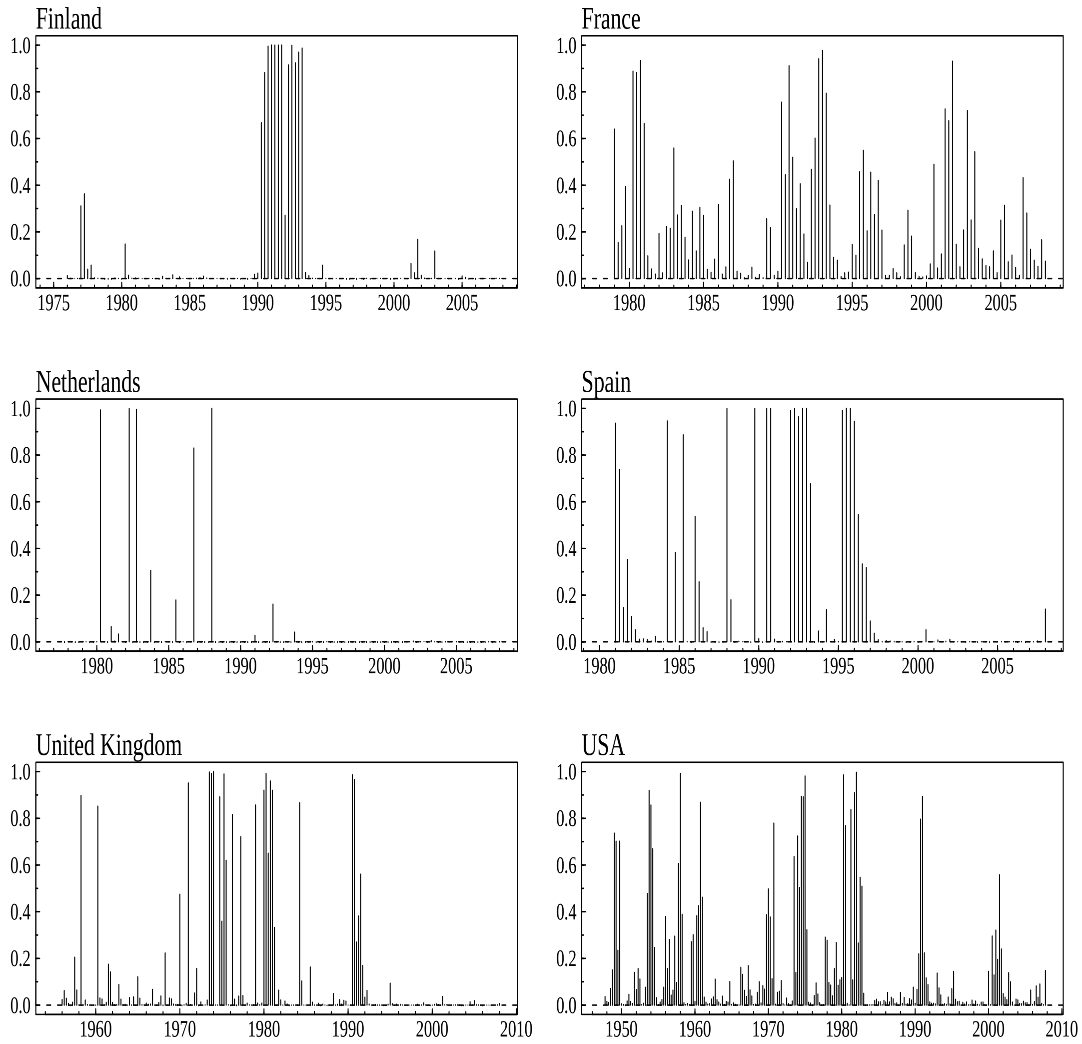

Figure 5: MS AR model. Fitted values

Based on results in Table 5. Dependent variable and fitted values in an AR model with a two-state Markov switching in the mean, equation (11). The dependent variable is 100 times the first differences of the logarithm of the GDP series, except for UK and US where a Box-Cox transformation with Box-Cox parameter 1 and 0.5, respectively for each series is applied.

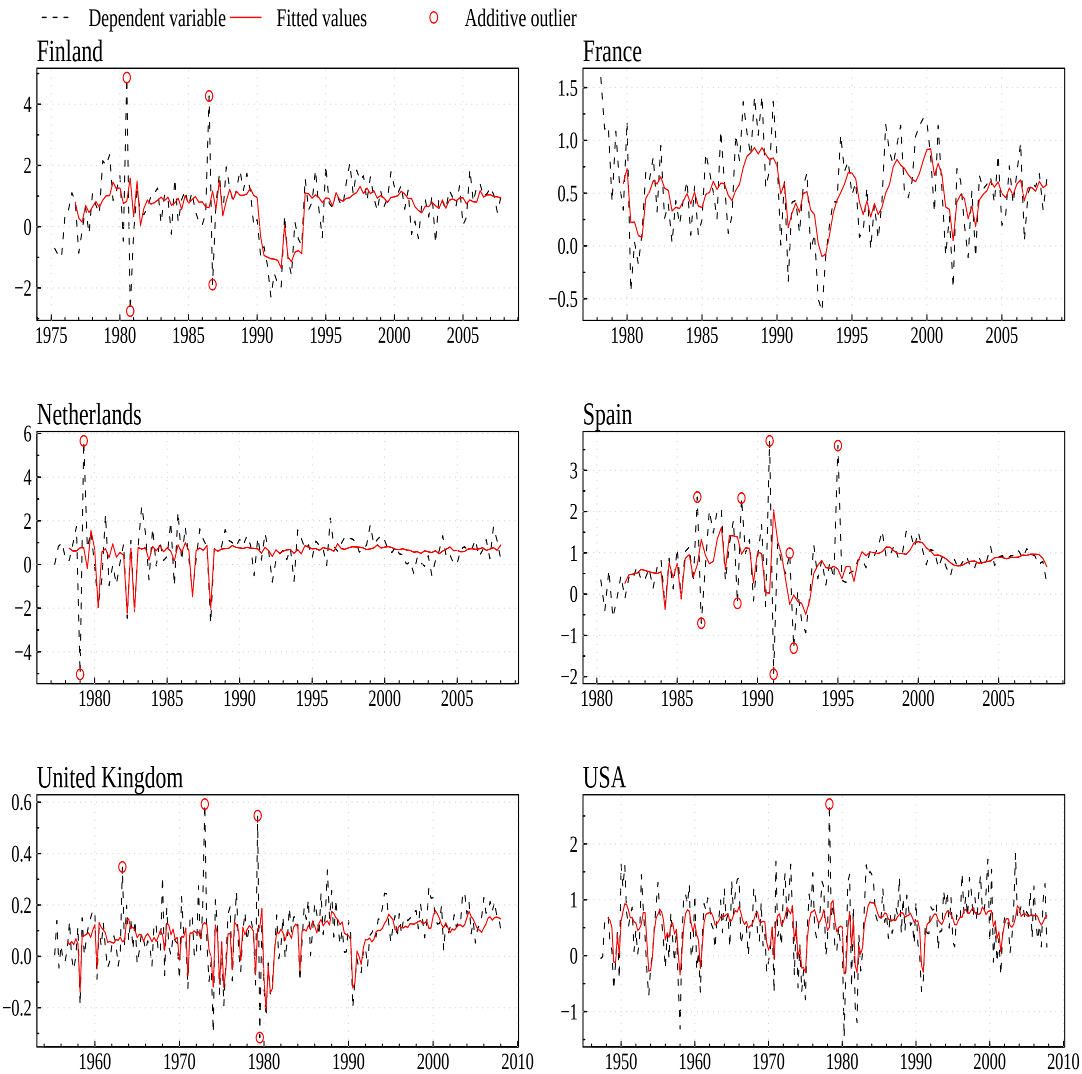

Figure 6: UK GDP. AR model with MS in mean and variance

Growth rates of the UK GDP series and filtered probabilities of regime 1 in an AR model for the growth rates with a two-state Markov switching in the variance. The figure suggests a structural change in the variance of the data.

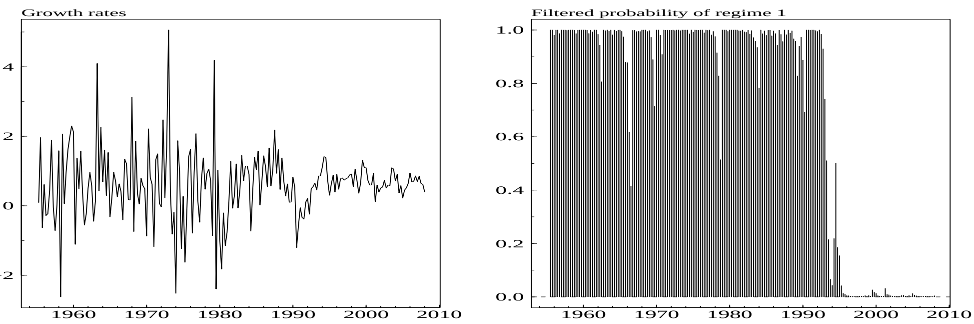

Figure 7: Structural Markov switching model with correlated components

Based on results reported in Table 7 for model 2 (equations (35)-(38)): trend with a switching variable plus transitory component. The input series is 100 times the logarithm of the US GDP. Official NBER trough dates are indicated with vertical dotted lines in the top graphic. They are reasonably matched in most of the cases, except for the recession at the beginning of 1970, where the second phase of the cycle is stretched compared to the official business cycle dates.

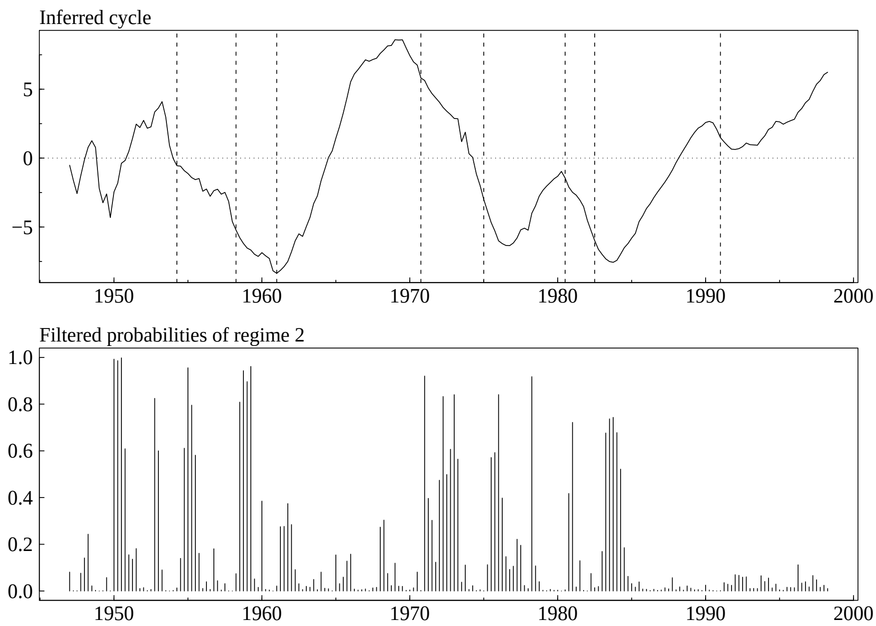

Figure 8: FR GDP series: Two transitory/cyclical components

The model given in equations (25)-(30) including a trend plus two transitory components in the trigonometric representation is estimated for the GDP of France. The periodogram depicts the spectral density estimate for the series after removing a linear trend to render stationarity. The trend and cycle components and the filtered probabilities are inferred from the estimated model reported in Table 8. Two cycles are detected. One of them is purely deterministic with periodicity 42 quarters. The second cycle is stochastic with periodicity 21 quarters. One cycle of the component of lower periodicity is completed within half cycle of the other component, approximately. The highest probabilities of regime 1 appear to coincide with a trough in the cycle of higher periodicity, especially at the beginning of the sample.